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Thermal diffusion in cyclic laminated composites: spectral properties and application to the homogenization

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Abstract—The eigenvalue problem associated with thermal diffusion studies in cyclic multilayered composites is considered. The transfer matrix of these walls can be made explicit with the help of Chebycheff's polynomials. This general property is exploited in a detailed study of the first group of eigenvalues in order to propose a homogenization process for these media. By this way it is possible to find a homogeneous medium which has approximately the same first eigenvalues as the actual multilayered composite. The application to a binary composite leads to an interpretation of the time scale of the homogenized medium. This example reveals a very good accuracy in the temperature calculations. A corrective term which takes into account the finite number of the pattern repetitions, when determining the physical characteristics of the homogenized wall, is at the origin of that remarkable accuracy.

1. INTRODUCTION

AMONG the numerous methods permitting us to solve the thermal diffusion problems in multilayered composites, the finite integral transform is an interesting candidate: it gives the analytical solution through a simple development; it is entirely built up in the physical space. Thus it simplifies the physical analysis of the phenomenon and it uses few numerical calculations. We know that it needs the solution of an intermediary Sturm–Liouville problem; but provided that a reliable technique is used for finding the eigenvalues, this essential step is led without any difficulty.

When the number of layers of the composite rises, the calculation of the spectrum becomes simply more lengthy. The achievement of the full solution of the thermal problem is thus more lengthy itself and thus is more time-consuming. The cyclic repetition of a given pattern does not change the nature of the last problem, but we must ask ourselves if such a periodicity is a source of simplification—notably when the number of patterns becomes large a homogenization ought to be practicable.

After a brief quote of the method, we build the characteristic equation of a cyclic composite medium. We focus then on the first eigenvalues and on the search of an approximate homogeneous medium which is able to give the same eigenvalue spectrum.

Throughout the article, a binary copper/glass composite supports the illustrations. This kind of association of a poor and a good conducting material which may give rise to industrial applications is not easier to deal with as we show later.

2. METHOD OF SOLUTION

Let us consider an N layered wall (Fig. 1). Each layer possesses its own constant thermophysical properties and thermal contact between the various layers is supposed to be perfect; the external faces are submitted to stationary conditions of the third kind. We consider a transitory thermal regime. Owing to the linearity, the temperature $T_i = T_i(x_i, t)$ is the sum of the stationary solution $S_i(x_i)$ and of the homogeneous transitory solution $\Theta_i = \Theta_i(x_i, t)$ which obeys the equation:

$$\frac{\partial \Theta_i}{\partial t} = \frac{\partial^2 \Theta_i}{\partial x_i^2}, \quad t > 0, \quad 0 < x_i < e_i, \quad i = 1, \ldots, N \tag{1}$$

with [1]:

$$t = t'/\tau', \quad x_i = \frac{x'_i}{e'_1}\left(\frac{a'_1}{a'_i}\right)^{1/2}, \quad e_i = \frac{e'_i}{e'_1}\left(\frac{a'_1}{a'_i}\right)^{1/2}.$$

The time scale τ' is here the time scale of the first layer

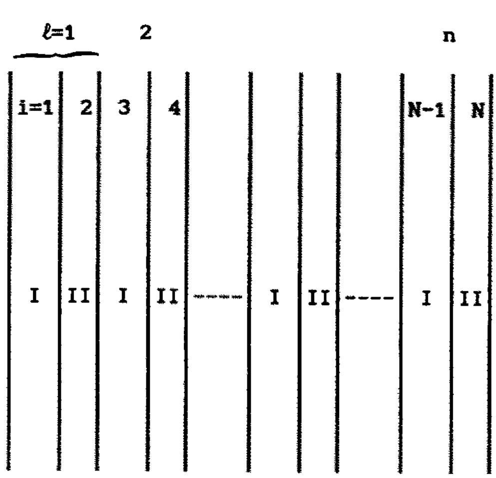

FIG. 1. Model of laminated binary composite. In all the succeeding numerical examples: I = copper, II = glass.

NOMENCLATURE

a, b, c, d elements of the pattern transfer matrix
a'_i thermal diffusivity
A^k kth coefficient of the development on the eigenbasis
b' thermal effusivity
C_i heat capacity
e'_i, (e_i) thickness of the ith layer (reduced thickness)
e_θ, e_ϕ reduced thicknesses of a homogenized wall
E_ϕ $= ne_\phi$
E_θ $= ne_\theta$
$F_i(x_i)$ flux density (in the eigenvalue problem)
h_i thermal surface conductance
I_λ, I^+, I^- intervals of definition of the $Z(\mu)$ function
L', (L) space scale (reduced space scale)
R_i thermal resistance of a layer
$[R_i]$ thermal resistance matrix
r_k eigenvalue
$S_i(x_i)$ stationary solution
t', (t) time (reduced time)
T_i, $T_i(x_i, t)$ dimensionless temperature in the ith layer
$[T]$ transfer matrix of the basic pattern
x'_i, (x_i) abscissa (reduced abscissa)
$X_i(x_i)$, $X_i^k(x_i)$ temperature (in the eigenvalue problem), eigenfunction
$Z(\mu)$ $= \frac{1}{2}\text{trace}[T]$.

Greek symbols
α_n, $\alpha_n(Z)$ $n-1$ order Chebycheff's polynomial of the second kind
β_i $= b'_i/b'_1$, reduced thermal effusivity
$[\gamma_i(\mu)]$ transfer matrix of the ith layer of a multilayered wall
$[\Gamma_N(\mu)]$ transfer matrix of an N layered wall
$\delta(\mu)$ $= (a-d)/2$
ε $= \frac{1}{2}((\beta_1/\beta_2)^{1/2} - (\beta_2/\beta_1)^{1/2})$
Θ_i, $\Theta_i(x_i, t)$ dimensionless temperature (in the homogeneous problem)
μ, μ_k, λ_k eigenvalue parameter, eigenvalues
ν parameter
ξ_N, η_N, ζ_N, χ_N coefficients of the transfer matrix of an N layered wall
τ' time scale
ω, ω_k, Ω_k eigenvalue parameter, eigenvalues.

Subscripts
i layer indice
h homogenized medium
k, l eigenvalue indice
n number of patterns in a cyclic composite
N number of layers in a multilayered composite
θ homogenized medium (given temperature case)
ϕ homogenized medium (given flux case).

$$\tau' = e_1'^2/a'_1.$$

Equation (1) possesses the general solution:

$$\Theta_i = X_i(x_i)\exp(-\mu^2 t), \quad 0 < x_i < e_i, \quad i = 1, \ldots, N \tag{2}$$

with

$$X_i(x_i) = X_i(0)\cos(\mu x_i) - F_i(0)\sin(\mu x_i)/\mu\beta_i$$

where $\beta_i = b'_i/b'_1$ is the reduced thermal effusivity of the layer i and $F_i(x_i)$ is the flux density.

By introducing the transfer matrix $[\gamma_i(\mu)]$ of each layer we may write:

$$\begin{bmatrix} X_i(e_i) \\ F_i(e_i) \end{bmatrix} = [\gamma_i(\mu)] \begin{bmatrix} X_i(0) \\ F_i(0) \end{bmatrix}, \quad i = 1, \ldots, N$$

with

$$[\gamma_i(\mu)] = \begin{bmatrix} \cos(\mu e_i) & -\sin(\mu e_i)/\mu\beta_i \\ \mu\beta_i \sin(\mu e_i) & \cos(\mu e_i) \end{bmatrix}.$$

The temperature and the flux density are continuous at the interfaces, and for the overall wall we have:

$$\begin{bmatrix} X_N(e_N) \\ F_N(e_N) \end{bmatrix} = [\Gamma_N(\mu)] \begin{bmatrix} X_1(0) \\ F_i(0) \end{bmatrix} \tag{3}$$

where $[\Gamma_N(\mu)]$, the transfer matrix of the wall, is the product:

$$[\Gamma_N(\mu)] = \begin{bmatrix} \xi_N(\mu) & \eta_N(\mu) \\ \zeta_N(\mu) & \chi_N(\mu) \end{bmatrix} \equiv \prod_{i=N}^{1} [\gamma_i(\mu)]. \tag{4}$$

Let

$$[R_1] = \begin{bmatrix} 1 & -1/h_1 \\ 0 & 1 \end{bmatrix} \quad \text{and}$$

$$[R_{N+1}] = \begin{bmatrix} 1 & -1/h_{N+1} \\ 0 & 1 \end{bmatrix}$$

be the thermal resistance matrices of both external faces.

By introducing the equilibrium temperatures $X_{N+1} = 0$, $X_0 = 0$ (homogeneous problem) of the ambient medium and the corresponding flux densities (which are equal to the flux densities crossing the external faces) F_{N+1} and F_0, we obtain the matrix relation

$$\begin{bmatrix} 0 \\ F_{N+1} \end{bmatrix} = [R_{N+1}][\Gamma_N(\mu)][R_1] \begin{bmatrix} 0 \\ F_0 \end{bmatrix}.$$

This system admits a nontrivial solution only if the superior off-diagonal coefficient of the above product of matrices is null. In a developed writing using equation (4) the equation reads:

$$-\xi_N(\mu)/h_1 + \zeta_N(\mu)/h_1 h_{N+1} + \eta_N(\mu) - \chi_N(\mu)/h_{N+1} = 0. \quad (5)$$

The roots of this equation are the eigenvalues, μ_k. We notice that a temperature given problem (on both external faces) is obtained when h_1 and h_{N+1} go to infinity and then equation (5) reduces to $\eta_N = 0$. In the same way a flux given condition (h_1 or $h_{N+1} = 0$) on one face and a given temperature condition on the other face give: $\xi_N = 0$ when the flux is given at $x_1 = 0$ or $\chi_N = 0$ when the flux is given at $x_N = e_N$. The eigenfunction $X_i^k(x_i)$, $i = 1, \ldots, N$ is associated with the corresponding eigenvalue μ_k. The eigenfunction basis is orthogonal—see Bouzidi and Duhamel [2] or Bouzidi [3], for example—and is used to develop the solution of the homogeneous problem:

$$\Theta_i(x_i, t) = \sum_k A^k X_i^k(x_i) \exp(-\mu_k^2 t).$$

The A^k coefficients are calculated by developing the initial given temperature field on the eigenbasis.

The key of the technique is the numerical calculation of the eigenvalues. Experience shows that, as soon as the number of layers reaches four or five, there exists a risk of missing some roots if a classical root-finding method is used. This is in particular the case for a cyclic composite. For instance, Fig. 2 gives the look of the function ξ_{2n} for a binary copper/glass composite with $n = 20$ repetitions of the basic binary pattern. The distribution of the zeros is surprising with a first interval]0, 0.08[where the roots lay more or less regularly spaced and beyond, some intervals where they pile up. This astonishing structure of the eigenvalue spectrum was noticed by Bouzidi [3] and is partly the motivation of this paper. It shows that a reliable root-finding method is necessary but we have built earlier such a safety method [4].

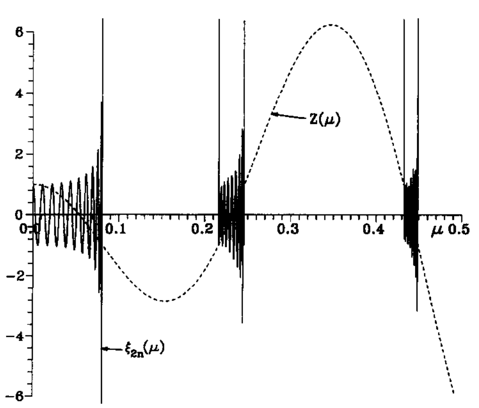

FIG. 2. Characteristic function $\xi_{2n}(\mu)$ for a binary composite (20 patterns). Given flux condition on the left-hand side.

3. TRANSFER MATRIX OF THE CYCLIC COMPOSITE-EIGENVALUE EQUATION

Let us consider a multilayered wall which is the repetition of a given pattern n times. The pattern itself is the assembly of 2 or 3 or... layers with distinct physical properties. Its transfer matrix is written as:

$$[T] \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with det $[T] = ad - bc = 1$ (passive medium). We set:

$$tr[T] = a + d = 2Z(\mu) \quad (6)$$

and

$$a - d = 2\delta(\mu). \quad (7)$$

The overall transfer matrix $[\Gamma_N(\mu)]$ (with now $N = 2n$ or $3n$ or...) is the nth power of $[T]$. The Hamilton–Cayley theorem permits us to write:

$$[T]^n = \alpha_n[T] + \alpha_n'[I] \quad (8)$$

where $[I]$ is the unit matrix and,

$$\alpha_n(Z) = 2Z\alpha_{n-1} - \alpha_{n-2} \quad (8a)$$

$$\alpha_n' = -\alpha_{n-1}. \quad (8b)$$

The recurrence is initialized by:

$$\alpha_1 = 1 \quad \text{and} \quad \alpha_2 = 2Z(\mu). \quad (8c)$$

We shall be able to calculate the four coefficients of the $[T]^n$ matrix, with the help of equation (8), as soon as the recurrence (8a–c) is solved.

Let us suppose that there exists intervals $I_\lambda = [\mu_i, \mu_m]$ such as

$$|Z(\mu)| \leqslant 1 \quad \text{when } \mu \in I_\lambda.$$

Then, in these intervals, the solution is the Chebycheff's polynomial of the second kind. We recall that by setting

$$Z(\mu) = \cos\theta$$

this polynomial reads

$$\alpha_n(\mu) = \frac{\sin(n\theta)}{\sin\theta}.$$

We notice that in the neighbourhood of the zeros, λ_k of $Z(\mu)$, necessarily the I_λ intervals exist. Outside the I_λ intervals the Chebycheff's polynomial may be extended. By defining I^+ and I^- intervals such as:

$$Z(\mu) > 1 \quad \text{when } \mu \in I^+ \quad \text{and}$$

$$Z(\mu) < -1 \quad \text{when } \mu \in I^-$$

we easily show that, when $\mu \in I^+$, the extension can be written as:

$$\alpha_n(Z) = \frac{\mathrm{sh}(n\theta)}{\mathrm{sh}\theta}, \quad \text{by setting } Z(\mu) = \mathrm{ch}\theta.$$

While, on the I^- intervals we have:

$$\alpha_n(Z) = (-1)^{n-1}\frac{\mathrm{sh}(n\theta)}{\mathrm{sh}\theta}, \quad \text{by setting } Z(\mu) = -\mathrm{ch}\theta.$$

We thus could always express α_n and formally calculate the $[T]^n$ matrix coefficients.

Owing to (8), the eigenvalue equation (5) now can gradient of mass fraction of vapour, we can then write:

$$\alpha_n(Z)\left[b(\mu)-\frac{a(\mu)}{h_1}-\frac{d(\mu)}{h_{N+1}}+\frac{c(\mu)}{h_1 h_{N+1}}\right] + \alpha_{n-1}(Z)\left[\frac{1}{h_1}+\frac{1}{h_{N+1}}\right] = 0. \quad (9)$$

The first square brackets term represents the characteristic equation of the pattern when submitted to the same boundary conditions as the whole wall. It is only when the external thermal resistances vanish that the eigenvalues of the pattern are among the eigenvalues of the complete composite, that is to say, when the boundary conditions are first kind conditions. In that case, the characteristic equation (9) reduces to:

$$\alpha_n(Z)b(Z) = 0. \quad (10)$$

In a more general way, the cyclic form of the medium is translated by the Chebycheff's polynomials $\alpha_n(Z)$ and $\alpha_{n-1}(Z)$. Although it is not the subject of the paper, one can deduce from the results of Appendix 2 that the properties of these polynomials when $|Z(\mu)| \leqslant 1$ are responsible for the piling up of the eigenvalues near the roots of $Z(\mu)$ as we observed in Fig. 2.

4. STUDY OF THE FIRST EIGENVALUES, HOMOGENIZATION

When the heat exchange of the composite with the external medium takes place through convection—or linearized radiation—the characteristic equation (5) depends on both surface conductances h_1 and h_{N+1} whose values are entirely outside the cyclic nature of the medium. It is thus difficult to study this case in a general way. It is the reason why we concentrate our attention on problems with boundary conditions of first and second kind which are 'purer'.

The first roots are the prevailing eigenvalues at the end of the thermal transient regime: they control long-term behaviour of the composite. Any one feels, furthermore, that as n becomes large, more exactly as e_i/n becomes small compared to the overall thickness, the composite must behave as a homogeneous medium.

4.1. *Imposed temperature case*

As μ goes to zero each individual transfer matrix of a multilayered wall goes to the thermal resistance matrix of the layer:

$$[\gamma_i(0)] = \begin{bmatrix} 1 & -e_i/\beta_i \\ 0 & 1 \end{bmatrix}, \quad \forall i.$$

Thus, whatever the number of layers of the pattern, we have

$$Z(0) = \tfrac{1}{2}tr[T] = 1 = \mathrm{Det}\,[T].$$

Furthermore we show in Appendix 1 that

$$\left.\frac{\mathrm{d}Z}{\mathrm{d}\mu}\right|_{\mu=0} = 0 \quad \text{and} \quad \left.\frac{\mathrm{d}^2Z}{\mathrm{d}\mu^2}\right|_{\mu=0} < 0.$$

The first I_λ kind interval is thus such that

$$0 \leqslant \mu \leqslant \mu_{\max} \quad \text{with } Z(\mu_{\max}) = -1.$$

We showed (Appendix 2) that it contains $(n-1)$ eigenvalues μ_k which obey the equation:

$$Z(\mu_k) = \cos(k\pi/n), \quad k = 1, 2, \ldots, (n-1).$$

Let us see if there exists a homogeneous medium, the first eigenvalues of which are approximations of the first eigenvalues of the true composite. Necessarily these values verify the characteristic equation of a homogeneous medium submitted to fixed temperature conditions:

$$\frac{\sin(\omega E_\theta)}{\omega\beta} = 0 \quad \text{or } \omega_k = k\pi/E_\theta, \quad k = 1, 2, \ldots$$

where E_θ is the adimensional thickness which is calculated with the same characteristic scale as that used for the actual composite. Thus E_θ must satisfy the condition:

$$\omega_k \to \mu_k \quad \text{when } k/n \ll 1$$

or

$$Z(k\pi/E_\theta) \to \cos(k\pi/n) \quad \text{when } k/n \ll 1. \quad (11)$$

Near zero, $Z(\omega)$ may be developed in Taylor's series:

$$Z(\omega) = 1 + \frac{\omega^2}{2!}\left.\frac{\mathrm{d}^2Z}{\mathrm{d}\omega^2}\right|_{\omega=0} + O(\omega^4)$$

and by choosing E_θ such as

$$\frac{1}{E_\theta^2}\left.\frac{\mathrm{d}^2Z}{\mathrm{d}\omega^2}\right|_{\omega=0} = -\frac{1}{n^2} \quad (12)$$

we may satisfy the condition (11) with an error of order $(k\pi/n)^4$.

Let us consider the preceding copper/glass binary composite. We have:

$$[T] \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$$a = \cos(\mu(e_1+e_2)) + \frac{\beta_2-\beta_1}{\beta_2}\sin(\mu e_1)\sin(\mu e_2),$$

$$b = -\frac{1}{\mu\beta_1}\left[\sin(\mu(e_1+e_2)) - \frac{\beta_2-\beta_1}{\beta_2}\cos(\mu e_1)\sin(\mu e_2)\right],$$

$$c = \mu\beta_2\left[\sin(\mu(e_1+e_2)) - \frac{\beta_2-\beta_1}{\beta_2}\cos(\mu e_1)\sin(\mu e_2)\right],$$

$$d = \cos(\mu(e_1+e_2)) - \frac{\beta_2-\beta_1}{\beta_1}\sin(\mu e_1)\sin(\mu e_2)$$

with

$$Z(\mu) = \tfrac{1}{2}tr[T] = (1+\varepsilon^2)\cos(\mu(e_1+e_2)) - \varepsilon^2\cos(\mu(e_1-e_2))$$

where

$$\varepsilon = \tfrac{1}{2}((\beta_1/\beta_2)^{1/2}-(\beta_2/\beta_1)^{1/2})$$

and

$$\left.\frac{d^2Z}{d\omega^2}\right|_{\omega=0} = -[(1+\varepsilon^2)(e_1+e_2)^2-\varepsilon^2(e_1-e_2)^2].$$

We deduce:

$$E_\theta^2 = n^2[(1+\varepsilon^2)(e_1+e_2)^2-\varepsilon^2(e_1-e_2)^2] = n^2e_\theta^2. \tag{13}$$

Remark: in the continuity of the preceding sections, the adimensional quantities are defined through the first layer reference scales (see Section 2), so as for the thicknesses. It is clear that this last scale, which is justified for pattern characterization, is not the best for the homogeneous medium reference scale. As we might expect the macroscopic adimensional thickness E_θ is thus n times the microscopic thickness e_θ due to that choice.

Figure 3 permits us to compare the first eigenvalues of the composite, which possesses $n = 40$ patterns, to those of the corresponding homogeneous medium (b_θ curve).† One may see that beyond the 12th eigenvalue, the estimates are no longer valid.

4.2. *Imposed flux on one face, imposed temperature on the other face*

The characteristic equation (9) may be written:

$$Z\alpha_n(Z)-\alpha_{n-1}(Z)+\nu\delta\alpha_n(Z) = 0 \tag{14}$$

with $\nu = 1$ if the flux is fixed on the left-hand side ($x_1 = 0$); $\nu = -1$ if the flux is fixed on the right-hand side ($x_N = e_N$). The first eigenvalues are obtained on the same I_λ interval as in the above section (see Appendix 2). Since $|Z| \leq 1$, we have

$$Z\alpha_n(Z)-\alpha_{n-1}(Z) = \cos(n\theta) = T_n(Z) \tag{15}$$

where $T_n(Z)$ is the Chebycheff's polynomial of the first kind, which has also the property:

$$\alpha_n(Z) = \frac{1}{n}\frac{dT_n}{dZ}.$$

Equation (14) can be written as:

$$T_n(Z)+\nu\frac{\delta}{n}\frac{dT_n}{dZ} = 0. \tag{16}$$

When μ goes to zero, δ/n is of order $O(\mu^2/n)$ and dT_n/dZ goes toward n^2. Thus as soon as μ^2n is small, equation (16) may be approximated by

$$T_n(Z) = 0. \tag{17}$$

But the left-hand side term is also the first order Taylor's development of $T_n[Z+\nu(\delta/n)]$, the equation

$$T_n\left(Z+\nu\frac{\delta}{n}\right) = 0 \tag{18}$$

is thus a better estimate as soon as μ^2n is small. The roots of equation (18) are such that

$$Z(\mu_k)+\nu\frac{\delta}{n}(\mu_k) = \cos\left(\frac{\pi}{2n}+k\frac{\pi}{n}\right),$$
$$k = 0,\dots,k_{\max} \quad \text{with } k_{\max} \leq (n-1). \tag{19}$$

In the same way as in the above section we may find an adimensional thickness E_ϕ such that the eigenvalues of a homogeneous medium submitted to a fixed temperature condition on one side and a fixed flux condition on the other approach the roots of equation (18). For the binary composite which supports the illustrations throughout this article, we find:

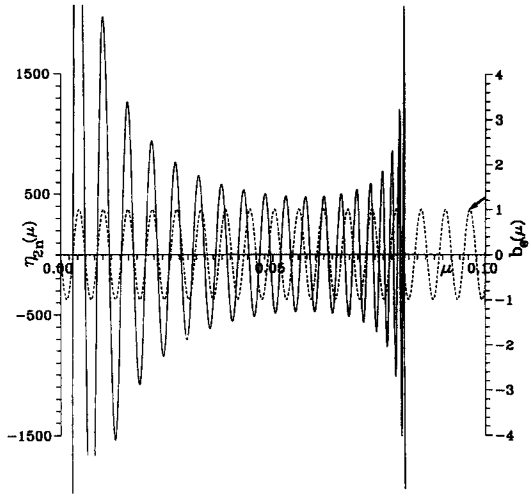

FIG. 3. Comparison of the actual characteristic function $\eta_{2n}(\mu)$ and of the homogenized wall characteristic function $b_\theta(\mu)$ (40 patterns). Imposed temperature case.

† We drew $b_\theta = -\sin(\mu E_\theta)$ as a matter of fact instead of $b_\theta = -\sin(\mu E_\theta)/\mu\beta$.

$$\delta = \varepsilon(1+\varepsilon^2)^{1/2}[\cos(\mu(e_1+e_2)) - \cos(\mu(e_1-e_2))]$$

and then

$$E_\phi^2 = n^2\left[\left(1+\varepsilon^2+\frac{v}{n}\varepsilon(1+\varepsilon^2)^{1/2}\right)(e_1+e_2)^2 - \left(\varepsilon^2+\frac{v}{n}\varepsilon(1+\varepsilon^2)^{1/2}\right)(e_1-e_2)^2\right] = n^2 e_\phi^2. \quad (20)$$

We notice that the microscopic reduced thicknesses e_ϕ and e_θ are related to each other by:

$$e_\phi^2 = e_\theta^2 + 4\frac{v}{n}\varepsilon(1+\varepsilon^2)^{1/2}e_1 e_2.$$

The microscopic reduced thickness of the fixed temperature problem, e_θ, is also suitable for the imposed flux problem, as we might expect since equation (17) is an approximation of the equation (16) it is the thickness when n go to infinity.

In Table 1 we may compare the first eigenvalues, r_k, of the binary copper/glass composite submitted to an imposed flux at $x_{2n} = e_{2n}$ ($v = -1$), to the roots of both equations:

$$\cos(\omega E_\theta) = 0 \quad \text{and} \quad \cos(\Omega E_\phi) = 0.$$

The composite has 40 patterns. The first Ω_k values are actually closer to r_k than are the ω_k values. For $k = 20$ neither estimate suits: Ω_{20} is as a matter of fact closer to r_{21} than it is to r_{20}.

5. PHYSICAL INTERPRETATIONS

The physical characteristics of the equivalent homogeneous medium may be calculated. When choosing the characteristic scales of the first layer of the pattern, a homogeneous medium of thickness L' and diffusivity a' have the reduced thickness

$$L = \frac{L'}{e'_1}\left(\frac{a'_1}{a'}\right)^{1/2}. \quad (21)$$

Thus, in the case of a binary composite of actual thickness $L' = n(e'_1+e'_2)$ and reduced thickness E_ϕ (equation (20)), we obtain from equation (21):

$$a' = \frac{(e'_1+e'_2)^2}{\dfrac{e'^2_1}{a'_1}+\dfrac{e'^2_2}{a'_2}+\dfrac{e'_1e'_2}{(a'_1a'_2)^{1/2}}\left[\left(1+\dfrac{v}{n}\right)\dfrac{b'_1}{b'_2}+\left(1-\dfrac{v}{n}\right)\dfrac{b'_2}{b'_1}\right]} \quad (22)$$

with

- $v = 0$: first kind problem
- $v = 1$: fixed flux on the left-hand side
- $v = -1$: fixed flux on the right-hand side.

When $v/n \to 0$, we find the same formula as Yamachita *et al.* [5], with a diffusivity which is independent of the chosen boundary conditions. In that case the calculated diffusivity preserves the thermal resistance—i.e. $n(e'_1/\lambda'_1+e'_2/\lambda'_2)$—and the heat capacity—i.e. $n(\rho'_1c'_1e'_1+\rho'_2c'_2e'_2)$—of the actual wall as advocated in refs. [6, 7].

The corresponding internal time scale is:

$$\tau'_h = (e'_1+e'_2)^2/a'$$

which may be written as:

$$\tau'_h = \tau'_1+\tau'_2+(\tau'_1\tau'_2)^{1/2}\left[\frac{b'_1}{b'_2}+\frac{b'_2}{b'_1}\right], \quad \text{when } v/n \to 0. \quad (23)$$

By introducing the thermal resistance R and the heat capacity, C, we know, Gosse [8], that there are two groups which characterize the diffusivity on a homogeneous medium: $RC = \tau'$, the internal time scale and $R/C = 1/b'^2$ which play a role in the surface time scale. In the same spirit, letting:

$$\tau'_{1/2} = R_1 C_2$$

and

$$\tau'_{2/1} = R_2 C_1$$

we may write the above time scale:

$$\tau'_h = \tau'_1+\tau'_2+(\tau'_1\tau'_2)^{1/2}[(\tau'_{2/1}/\tau'_{1/2})^{1/2} + (\tau'_{1/2}/\tau'_{2/1})^{1/2}]. \quad (24)$$

In a cyclic composite medium, the properties controlling the surface diffusion (that is to say the cross-time scales $\tau'_{1/2}$ and $\tau'_{2/1}$) give a contribution to the internal time scale as expected.

Table 1. Comparison of the first eigenvalues: r_k, actual composite, Ω_k, homogenized medium ($n = 40$ patterns), ω_k, homogenized medium ($n \to \infty$)

k	1	2	5	10	20
$\omega_k \times 10^3$	1.43794	4.3138	12.941	27.321	56.079
$\Omega_k \times 10^3$	1.45097	4.3529	13.006	27.568	56.588
$r_k \times 10^3$	1.45087	4.3514	13.023	27.232	53.457

6. COMPARISON BETWEEN THE HOMOGENEOUS AND THE ACTUAL MEDIUM

We give the results of this comparison for the case of a given flux on one face; we saw that this problem was more difficult to homogenize than the imposed temperature case.

Let us consider walls of physical thickness E' submitted to a given temperature at $x_1 = 0$ ($T(0, t) = 1$) and to a given flux (we chose a null flux condition) at $x_{2n} = e_{2n}$. We present in Fig. 4 the temperature value at $x_{2n} = e_{2n}$ as a function of the pattern repetition number (the pattern is the previous two layered copper/glass wall) and for different instants, knowing that, at the initial time, the whole wall is at zero temperature. There is no single meaningful time scale in these walls because as the number of patterns varies, the proper time scale of a given fixed n wall changes. Considering arbitrarily a thickness E' of 10 cm, the

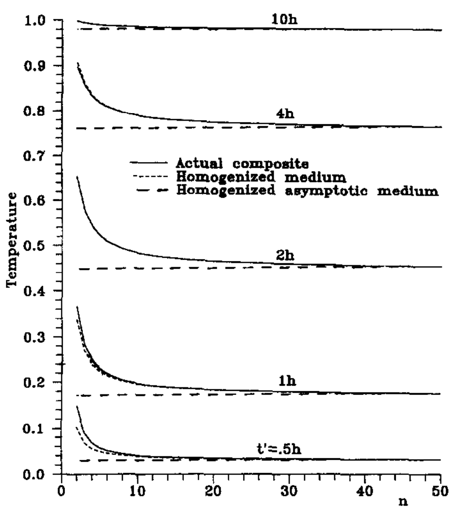

FIG. 4. Comparison of the actual wall and of the homogenized wall: temperature of the right-hand face as a function of the pattern repetition number. Null flux condition on the right-hand side.

time scale of the equivalent homogeneous medium, in the sense of Section 4.2, varies from 3.8 h when the wall has two patterns to 5.9 h when the wall is infinitely layered (equation (23)). For this reason, we do not use a reduced time when drawing Fig. 4 or 5.

We notice that for 50 repetitions, the multilayered wall may be classed as a homogenized asymptotic medium ($v/n \to 0$ in equation (22)) since the difference between the true temperature (continuous line) and the temperature of the homogeneous infinitely layered medium (asymptotic value) is less than 0.7% of the full scale whatever the considered instant is. At the opposite when we consider the wall made of two patterns the gap between the true temperature and the

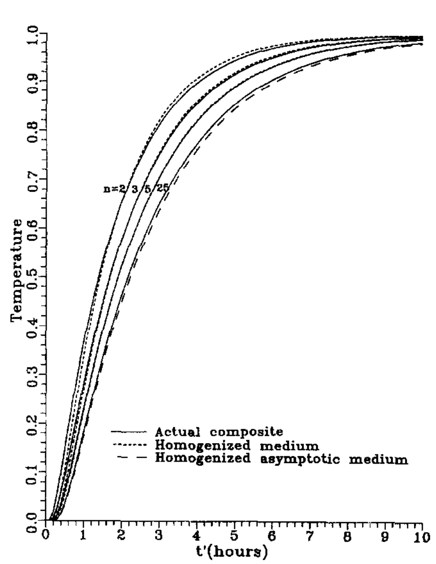

FIG. 5. Comparison of the actual wall and of the homogenized walls.

asymptote may exceed 20%, which is not a surprising observation.

When using the thermophysical homogeneous characteristics which suit the imposed flux studied problem (equation (22) with $v = -1$), we see that the corrective term which takes into account the finite number of layers produces a remarkable improvement. The temperature results appear in dotted lines. Even when the wall has only two patterns, the error in temperature forecast is always less than 3% (full scale reference), whatever the time in the studied range. Thanks to this corrective term, the medium becomes homogenizable from 10 repetitions (for $t' = 0.5$ h and beyond); it may even be correctly homogenized from 3 repetitions for all the instants more than 1 h (i.e. 22% of the proper time scale of the 3 pattern wall).

Figure 5, which shows the rise of the temperature during the transitory regime at the $x_{2n} = e_{2n}$ face of the composite, confirms the interest of the corrective term when the number of pattern repetitions is finite. It is only for the short-term behaviour—i.e. for instants less than 15% of the time-scale—and for a medium which is made of only 2 patterns—that this corrective term loses its efficiency because of errors higher than 4% (full scale reference). One knows that, for these short times, the construction of the solution requires a great number of terms of the eigenbasis development; as a matter of fact less than $n/2$ eigenvalues of the true problem are correctly approximated through homogenization (see Section 4) of an n pattern wall; thus one might expect the impossibility of short-term description of media which possess few patterns.

7. CONCLUSION

The study of the eigenvalue equations stemming from diffusion problems in cyclic multilayered walls permits us to bring forward common features of the eigenvalue spectrum of these media. We showed that the equation is made explicit with the help of Chebycheff's polynomials. The properties of these polynomials induce common characteristics such as the grouping of eigenvalues round the zeros of the trace of the transfer matrix of the basic pattern.

The first set of eigenvalues plays a fundamental role in the long-term thermal behaviour of the composite. By studying this first set, we showed that there exists a homogeneous medium, the first eigenvalues of which approach those of the actual composite and we proposed several approximations of the physical characteristics of this medium; one of these approximated properties takes into account the finite number of patterns.

The example of a binary composite permits us to show how the time scale of the homogenized medium brings into play not only both internal time scales of the constitutive layers but also two symmetrical cross-scales which characterize the interface diffusion. The comparison of temperature calculation results show that it is possible to correctly homogenize cyclic multilayered walls which possess a small number (3–4) of patterns thanks to the corrective term which takes into account the finite number of patterns.

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APPENDIX 1

Let us consider a pattern of p different layers. Its transfer matrix $[T]$ is the product of the transfer matrices $[\gamma_i(\mu)]$ of layers (see Section 2.1). Each coefficient of the $[\gamma_i(\mu)]$ matrix is even referring to μ: the coefficients of the product are thus even and so is every linear function of these coefficients; so is trace $[T]$ in particular.

Thus

$$\left.\frac{dZ}{d\mu}\right|_{\mu=0} = 0.$$

Near $\mu = 0$, $[\gamma_i(\mu)]$ takes the form:

$$[\gamma_i(\mu)] = [R_i] - \frac{\mu^2}{2} e_i^2 [U_i] + \cdots$$

where $[R_i]$ is the thermal resistance matrix of the layer and where

$$[U_i] = \begin{bmatrix} 1 & -e_i/3\beta_i \\ -2\beta_i/e_i & 1 \end{bmatrix}.$$

Near $\mu = 0$, we have thus:

$$[T] = \prod_{i=p}^{1} [R_i] - \frac{\mu^2}{2} \sum_{i=1}^{p} e_i^2 \left(\prod_{k=p}^{i+1} [R_k] \right) [U_i] \left(\prod_{k=i-1}^{1} [R_k] \right). \quad \text{(A1)}$$

But,

$$\prod_{k=n}^{m} [R_k] = \begin{bmatrix} 1 & -\sum_m^n e_k/\beta_k \\ 0 & 1 \end{bmatrix} \quad \text{for } n \geqslant m \quad \text{and } [I] \text{ for } n \leqslant m.$$

The coefficients of the first diagonal of the product matrix which appears in the second term of (A1) take the form:

$$1 + 2\frac{\beta_i}{e_i} \sum_p^{i+1} \frac{e_k}{\beta_k} \quad \text{for the first coefficient,}$$

$$1 + 2\frac{\beta_i}{e_i} \sum_{i-1}^{1} \frac{e_k}{\beta_k} \quad \text{for the second coefficient.}$$

We deduce:

$$Z \equiv \tfrac{1}{2} tr[T] = 1 - \frac{\mu^2}{2} \sum_{i=1}^{p} e_i \beta_i \sum_{k=1}^{p} \frac{e_k}{\beta_k} + \cdots$$

which shows that

$$\left.\frac{d^2 Z}{d\mu^2}\right|_{\mu=0} < 0.$$

APPENDIX 2

We recall a result of Rayleigh's theorem [9]:

If the eigenvalues μ_k of a given boundary linear problem are arranged in ascending order of magnitude, the eigenvalues λ_k of the problem in which one supplementary condition—here a given temperature at an interface—is imposed upon the medium are such that,

$$\lambda_{k-1} \leqslant \mu_k \leqslant \lambda_k \quad k = 1, 2, \ldots.$$

The theorem was originally proved for linear vibration problems but it is easy to show [4] that the Sturm–Liouville problem arising from the thermal diffusion in layered composites is a special type of free vibration problems.

Let us consider a cyclic composite wall and a given temperature problem for this wall. When all the interfaces between the n basic patterns are at a fixed temperature (constrained state), each eigenvalue λ_k of the pattern possesses a multiplicity of order n. At each time one interface constraint is removed, the multiplicity order diminishes by one unit while one novel eigenvalue of magnitude lower than λ_k appears on the μ-axis. When all the interface constraints are removed, the eigenvalue of order $k+n-1$ lies at the λ_k position—it is a single root of the final eigenvalue equation—though there are $n-1$ other roots lying between λ_{k-1} and λ_k in general.

When $k = 1$, there are $n-1$ eigenvalues between 0 and λ_1. The nth eigenvalue is $\mu_n = \lambda_1$, knowing that λ_1 is the first eigenvalue of the pattern when submitted to imposed temperatures on both its faces. We showed, in Section 3, that, for the considered case, the eigenvalue equation can be written as:

$$\alpha_n(Z) b(\mu) = 0.$$

The roots of $b(\mu)$ are the eigenvalues, λ_k, of the pattern (see equation (5) when the thermal resistances $1/h_1$ and $1/h_{N+1}$ vanish). The other roots which are solutions of

$$\alpha_n(Z) = 0$$

appear when $|Z| \leqslant 1$.† They are the roots of the Chebycheff's polynomial:

$$Z(\mu_l) = \cos(l\pi/n) \quad l = 1, 2, \ldots (n-1).$$

In particular, there are $n-1$ eigenvalues of this kind between 0 and λ_1. The first I_i interval:

$$[0, \mu_{\max}] \quad \text{with } \mu_{\max} < \lambda_1$$

contains the first $n-1$ eigenvalues.

† The equation $sh(n\theta)/sh(\theta) = 0$ has no solution.

Let us consider now a second kind problem. We saw in Section 2 that, by introducing two fictitious layers, with a constant transfer matrix (the thermal resistance matrices $[R_1]$ and $[R_{N+1}]$) at both sides of a multilayered wall, every problem is solved as a first kind problem: the eigenvalue equation is obtained by setting at zero the first off-diagonal coefficient of the overall transfer matrix. We may thus extend the above reasoning to second kind problems.

For instance, if the temperature is given on the left-hand side of the wall and the flux is given on the other side, we conclude that the eigenvalues μ_k of the problem which possesses the last constraint $X_N(e_N) = 0$—this is the previously studied first kind problem—and the calculated eigenvalues r_k (solutions of a problem in which the last constraint is removed) are such that:

$$\mu_{k-1} \leqslant r_k \leqslant \mu_k$$

where the μ_ks have the previous properties; this leads to:

$$0 < r_1 \leqslant \mu_1 \cdots$$

$$r_{n-1} \leqslant \mu_{n-1} \leqslant r_n \leqslant \mu_n\,(\equiv \lambda_1) \leqslant \cdots$$

$$r_{\ln-1} \leqslant \mu_{\ln-1} \leqslant r_{\ln} \leqslant \mu_{\ln}(\equiv \lambda_l) \leqslant \cdots.$$

The first I_λ interval $[0, \mu_{max}]$ with $\mu_{max} < \lambda_1$ contains at least the first $n-1$ eigenvalues. There is one eigenvalue between μ_{n-1} and λ_1. We cannot easily derive a conclusion on the membership of the nth root of the first I_λ or I^- interval because the eigenvalue equation of second kind problems is not so easily tractable as the first kind eigenvalue equation. Nevertheless a discussion should be possible.